Quasiconformal deformations associated to a wandering component.

Baker's proposition only allows to simplify original Sullivan's argument. It consists on constructing, by means of quasiconformal deformations, a space of rational maps of degree d of very big dimension (using of course the existence of a wandering component), bigger than the actual dimension of $Rat_d(\hat{\mathbb{C}})$, obtaining a contradiction.

Here we see how to construct quasiconformal deformations associated to wandering components.

Let f be a rational map in $\hat{\mathbb{C}}$ of degree $d \geq 2$.

Up to conjugacy by an element of $Aut(\hat{\mathbb{C}})$, we may assume that $f(\infty)=\infty$, i.e.,
$$f(z) = \frac{c_0 + \dots + c_{d-1} z^{d-1} + z^d}{c_d + \dots + c_{2d-1} z^{d-1}}.$$

Let Ω be a simply connected wandering component, so that $f^n: \Omega \to \Omega_n$ is a biholomorphism $\forall n \in \mathbb{N}$, as given by Baker's proposition.

A quasiconformal deformation is a rational function $\tilde{f}: \hat{\mathbb{C}} \circlearrowleft$ obtained by conjugating f by a quasi-conformal homeomorphism $\Phi: \hat{\mathbb{C}} \circlearrowleft$.

This corresponds to constructing a Beltrami coefficient μ, and taking the corresponding normalised quasiconformal map Φ_μ

Let $\mu \in L^\infty(\Omega)$, $\|\mu\|_\infty < 1$ be a Beltrami coefficient.

We can extend μ on $\hat{\mathbb{C}}$ as follows.

First, we may extend μ to the forward orbit $\overset{\mathcal{O}_f(\Omega_0)}{\bigcup \Omega_n}$ of Ω_0 by setting

$$\mu(z) = g_n^* \mu(z) \left(= (f^n)_* \mu(z)\right) = \mu(g_n(z)) \cdot \frac{\overline{g_n'(z)}}{g_n'(z)}, \; \forall z \in \Omega_n, \quad = \frac{\overline{(f^n)'(g_n(z))}}{(f^n)'(g_n(z))}$$

where g_n is the inverse of $f^n|_{\Omega_0} : \Omega_0 \to \Omega_n$ ← using Ω_0 wandering

$\forall z \in \bigcup \Omega_n \; \exists! \, n, \; z \in \Omega_n$

Then, we may extend μ to the preimages $f^{-k} \mathcal{O}_f(\Omega_0)$ by setting:

$$\mu(z) = (f^k)^* \mu(z) = \mu(f^k(z)) \cdot \frac{\overline{(f^k)'(z)}}{(f^k)'(z)} \quad \forall z \in f^{-k}(\Omega_n).$$

Hence μ is defined on the totally invariant set $G\mathcal{O}_f(\Omega)$.

Its complementary is also totally invariant, and we set $\mu \equiv 0$ on $\hat{\mathbb{C}} \setminus G\mathcal{O}_f(\Omega)$.

We have hence extended any Beltrami coefficient in Ω to a Beltrami coefficient (always denoted μ) on $\hat{\mathbb{C}}$, which is, by construction, f-invariant: $f^* \mu = \mu$.

By the Measurable Riemann mapping theorem (Ahlfors-Bers theorem), $\exists!$ Φ_μ normalized quasi ~~conformal~~ homeomorphism of $\hat{\mathbb{C}}$ satisfying the Beltrami equation.

Moreover, since $f^* \mu = \mu$, the map $f_\mu = \Phi_\mu \circ f \circ \Phi_\mu^{-1}$ is a rational map, which is the quasiconformal deformation of f associated to μ.

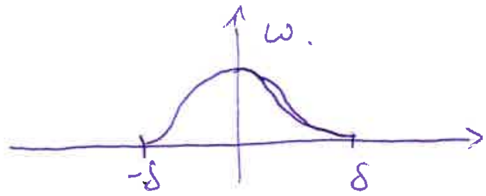
Generate an arbitrarily large space of quasiconformal deformations

We want to apply here the previous construction of quasiconformal deformations to a large class of Beltrami coefficients in Ω.

First, let $\delta > 0$, and set: $\omega : \mathbb{R} \to \mathbb{R}$ given by.

$$\omega(x) = \begin{cases} \delta^2 e^{\frac{\delta^2}{x^2-\delta^2}} & |x|<\delta \\ 0 & |x| \geq \delta \end{cases}$$

ω is a $\mathcal{C}^\infty$ map, with $0 \leq \omega \leq \frac{\delta^2}{e}$ and $|\omega'| \leq \frac{8}{e^2}\delta$.



Let now $m \in \mathbb{N}$ be any integer, and $0 = \alpha_0 < \alpha_1 < \dots < \alpha_{m+1} = \pi$ be a partition of $[0, \pi]$.

For any $b = (b_1 \dots b_m) \in Q_m := \{ b \in \mathbb{R}^m : |b_j| < 1 \} = (-1,1)^m$, we define:

$$S(\theta, b) := \sum_{j=1}^{m} b_j\, \omega(\theta - \alpha_j) \qquad (\theta \in [0, 2\pi))$$

If $\delta < \frac{1}{2} \min |\alpha_{j+1} - \alpha_j|$, then $\left|\frac{\partial S}{\partial \theta}\right| \leq \sum_j |b_j| |\omega'(\theta - \alpha_j)|$

The condition ensures that $\omega'(\theta - \alpha_j) = 0$ for all but one index j. We get then: $\left|\frac{\partial S}{\partial \theta}\right| \leq \cancel{\max} |b_j| \cdot \frac{8}{e^2}\delta = \frac{8}{e^2}\delta \leq 2\delta$.

We set $\Psi_b(z) := z\, e^{iS(\arg z, b)}$ $z \in \mathbb{D}$, $\arg z \in [0, 2\pi)$.

$\Psi_b : \mathbb{D} \to \mathbb{D}$ defines a family of homeomorphisms of the unit disk, since, if we assume $\delta < \frac{1}{2}$, then $\left|\frac{\partial S}{\partial \theta}\right| < 1$

In fact, a geometric interpretation of Ψ_b is as follows:

take a ray $\rho e^{i\theta}$. Then it is sent to $\rho e^{i(\theta + S(b,\theta))}$, and $\left|\frac{\partial S}{\partial \theta}\right| < 1$ gives the invertibility of this map

rotation

10.3

Notice that $\forall \theta > \pi$, with our assumption on δ, we have that $\omega(\theta - a_j) = 0 \ \forall j$, and hence $S(1,\theta) = 0$, and Ψ_t is the identity on $\mathbb{D} \cap \{\operatorname{Im} z < 0\}$. Similarly, $\Psi_0 = \mathrm{Id}$, while $\forall t \neq 0$, $\Psi_t \neq \mathrm{Id}$.

Rem: In fact, the value of $S(a_j, t)$ is $t_j \cdot \frac{\delta^2}{e}$, and if $t \neq t'$, then $\Psi_t \neq \Psi_{t'}$.

We can compute the Beltrami coefficient of Ψ_t directly, and obtain

$$\nu_t(z) = \frac{-e^{2i\theta} S_\theta}{2 + S_\theta} \qquad \text{where } z = \rho e^{i\theta}$$

In fact, $\Psi_t(z) = z\, e^{iS(\theta)}$, $\theta = \theta(z,\bar z) = \frac{1}{2i}\left(\log z - \log \bar z\right)$

$$\Rightarrow \frac{\partial \Psi_t}{\partial \bar z} = z \cdot i S_\theta\, e^{iS(\theta)} \cdot \left(-\frac{1}{2i\bar z}\right) = -\frac{S_\theta}{2}\, e^{iS(\theta)} \cdot \frac{z}{\bar z} \qquad \left(S_\theta = \frac{\partial S}{\partial \theta}\right)$$

$$\frac{\partial \Psi_t}{\partial z} = e^{iS(\theta)} + z \cdot i S_\theta \cdot e^{iS(\theta)} \frac{1}{2iz} = e^{iS(\theta)}\left(1 + \frac{S_\theta}{2}\right)$$

$$\Rightarrow \nu_t = \frac{\frac{z}{\bar z}\left(-\frac{S_\theta}{2}\right)}{1 + \frac{S_\theta}{2}} = -\frac{e^{2i\theta} S_\theta}{2 + S_\theta}.$$

In particular, $\|\nu_t\|_\infty \leq \frac{2\delta}{2 - 2\delta} \to 0$ when $\delta \to 0$, and Ψ_t is quasi-conformal in this case.

Let now $\Phi : \Omega \to \mathbb{D}$ be a biholomorphism. We define $\mu_t := \Phi^* \nu_t$.

With the procedure described above, we extend μ_t to a Beltrami coefficient on $\hat{\mathbb{C}}$ which is f-invariant and hence induces a quasiconformal deformation $f_t = \Phi_{\mu_t} \circ f \circ \Phi_{\mu_t}^{-1}$.

We claim that these deformations are not trivial:

Proposition: For all ~~...~~ analytic curve $\gamma : [0,1] \to Q_m$ there exists $s \in (0,1]$ such that $f_{\gamma(s)} \neq f_{\gamma(0)}$. (I.e., $t \mapsto f_t$ is not constant along any analytic curve).

Conclusion of Sullivan's theorem:

Since $f(\infty) = \infty$, and Φ_{μ_t} are normalised, we get that $f_t(\infty) = \infty$.

We can hence represent the family f_t as

$$f_t(z) = \frac{C_0(t) + \dots + C_{d-1}(t) z^{d-1} + z^d}{C_d(t) + \dots + C_{2d-1}(t) z^{d-1}},$$

By construction, μ_t depends $\mathbb{R}$-analytically on the parameter $t \in Q_m$, and hence so as for Φ_{μ_t} and f_t. (~~...~~)

In particular $f_t^{-1}(0)$, $f_t^{-1}(\infty)$ and $f_t^{-1}(1)$ vary real analytically on t, and

$$\Gamma : Q_m \to \mathbb{C}^{2d}$$
$$t \mapsto \Gamma(t) = (C_0(t), \dots, C_{2d-1}(t))$$ is also real analytic

Take $m > 4d = \dim_{\mathbb{R}} \mathbb{C}^{2d}$. Hence the maximal rank k of $\left(\frac{\partial \Gamma}{\partial t}\right)$ is $k \leq 4d < m$. If t_0 is a value where the rank of $\left(\frac{\partial \Gamma}{\partial t}(t_0)\right)$ is k, then it is k on a neighborhood of t_0 (rank is ~~...~~ lower semicontinuous),

By the implicit function theorem, there exist a manifold M of dimension $m-k>0$ passing through t_0 where $\Gamma(t) \equiv \Gamma(t_0)$ $\forall t \in M$.

This contradicts the triviality of the family f_t, since for any analytic curve $\gamma \subset M$, we would have $\Gamma(t) = \Gamma(t_0)$, i.e.,

$f_t = f_{t_0}$.

This concludes the proof, provided that we prove the non triviality of the family of deformations $(f_t)_{t \in Q_m}$.

Proof of the non triviality of $(f_b)_{b\in Q_m}$ (10,11)

We proceed by contradiction, and assume there exists a analytic curve $\gamma:[0,1]\to Q_m$ such that $f_{\gamma(s)} = f_{\gamma(0)} =: \tilde{f}$ $\forall s\in[0,1]$.

Set $H_s := \Phi_{\gamma(0)} \circ \Phi_{\gamma(s)}^{-1}$.

Recall: $f_t = \Phi_{\mu_t} \circ f \circ \Phi_{\mu_t}^{-1} \rightsquigarrow f_{\gamma(s)} = \Phi_{\gamma(s)} \circ f \circ \Phi_{\gamma(s)}^{-1}$

Lemma: We have:

1) $H_0 = \mathrm{Id}$, $H_s \circ \tilde{f} = \tilde{f} \circ H_s$ $\forall s\in[0,1]$

2) $H_s(z) = z$ $\forall z\in \mathcal{J}(\tilde{f})$

3) $\forall V\subset \mathcal{F}(\tilde{f})$ Fatou component is H_s-invariant: $H_s(V) = V$ $\forall s\in[0,1]$

Proof: (1) comes directly from the construction.

~~Let~~ $\forall m\geq 1$, consider $\mathrm{Fix}(\tilde{f}^m) =: F_m$

Since $\tilde{f}^m$ commutes with H_s, we get $H_s(F_m)\subseteq F_m$, and actually $H_s(F_m) = F_m$ (applying the same argument to H_s^{-1})

Being $H_0 = \mathrm{Id}$, and $s\mapsto H_s(z)$ continuous for any fixed z, and $\mathrm{Fix}(\tilde{f}^m)$ discrete, we obtain that H_s must fix any point of $\mathrm{Fix}(f^m)$.

Since repelling cycles are dense in $\mathcal{J}(\tilde{f})$, H_s fixes $\mathcal{J}(\tilde{f})$ pointwise (i.e., $H_s|_{\mathcal{J}(\tilde{f})} = \mathrm{Id}$)

Similarly, H_s acts on $\mathcal{F}(\tilde{f})$ as continuous deformations of the identity, hence they preserve every connected component. $\square$

Consider now U a simply connected Fatou component and $\tilde{\varphi}: U\to \mathbb{D}$ a biholomorphism. If $\tilde{\varphi}$ extended to the boundary as an homeomorphism, then $G_s = \tilde{\varphi}\circ H_s \circ \tilde{\varphi}^{-1}: \mathbb{D}\to\mathbb{D}$ would extend to the boundary $\partial\mathbb{D}$, being the identity on $\partial\mathbb{D}$ (since $H_s|_{\mathcal{J}(f)} = \mathrm{Id}$).

It turns out that this conclusion still holds even without the control on $\tilde{\varphi}$.

Lemma (Becker) For $s \in [0,1]$, G_s extends to $\partial \mathbb{D}$, ~~and~~ as the identity.

Assume this is true, we conclude the proof.

We apply the lemma to $U = \Phi_{\gamma(0)}(\Omega)$, Ω the wandering Fatou component.

Since $\Phi_{\gamma(s)} = H_s^{-1} \circ \Phi_{\gamma(0)}$ and H_s^{-1} fixes all Fatou components ~~(...)~~ (and in particular U), then $\Phi_{\gamma(s)} : \Omega \to U$ $\forall s \in [0,1]$

$\varphi : \Omega \to \mathbb{D}$ used in the construction of μ_t.

Consider as before $\tilde{\varphi} : U \xrightarrow{\sim} \mathbb{D}$, and $g_s = \tilde{\varphi} \circ \Phi_{\gamma(s)} \circ \varphi^{-1} = \tilde{\varphi} \circ H_s^{-1} \circ \Phi_{\gamma(0)} \circ \varphi^{-1}$.

$$\begin{tikzcd}
\Phi_t \circlearrowright \Omega \arrow[r, "\varphi"] \arrow[d, bend right, "\Phi_{\gamma(s)}"'] \arrow[d, "\Phi_{\gamma(0)}"] & \mathbb{D} \circlearrowleft \Psi_t \arrow[d, bend left, "g_s"] \\
U \arrow[r, "\tilde{\varphi}"] & \mathbb{D}
\end{tikzcd}$$

(Beltrami coefficient)

The complex dilatation of g_s is:

$$\varphi_* \Phi_{\gamma(s)}^* \tilde{\varphi}^*(0) = \varphi_* \Phi_{\gamma(s)}^* = \varphi_* \mu_{\gamma(s)} = \nu_{\gamma(s)},$$

which is the Beltrami coefficient of $\Psi_{\gamma(s)}$

Hence there exists a conformal automorphism $\alpha_s : \mathbb{D} \circlearrowleft$ so that $g_s = \alpha_s \cdot \Psi_{\gamma(s)}$.

~~Becker~~ ~~of~~ The action of g_s on $\partial\mathbb{D}$, does not depend on s, since

$$g_0 \circ g_s^{-1} = \tilde{\varphi} \circ \Phi_{\gamma(0)} \circ \varphi^{-1} \circ \varphi \circ \Phi_{\gamma(s)}^{-1} \circ \tilde{\varphi}^{-1} = \tilde{\varphi} \circ H_s \circ \tilde{\varphi}^{-1} = G_s, \text{ which}$$

is the identity on $\partial \mathbb{D}$ by Becker's lemma.

By construction, $\Psi_{\gamma(s)}(e^{i\theta}) = e^{i\theta}$ $\forall \theta \in [\pi, 2\pi]$. In particular the action of $\alpha_0 \circ \alpha_s^{-1}$ on $\partial\mathbb{D} \cap \{\operatorname{Im} z \leq 0\}$ is the identity, which implies that $\alpha_s \equiv \alpha_0$ $\forall s \in [0,1]$. It follows that $\Psi_{\gamma(s)} = \alpha^{-1} \circ g_s$ restricted to $\partial\mathbb{D}$ does not depend on s, which is a contradiction with respect to the construction of Ψ_t (the value on $\partial\mathbb{D}$ determines t, see the remark). $\square$